PERGAMON

International Journal of Solids and Structures 38 (2001) 2249–2266

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Bounds to shakedown loads

Nestor Zouain *, José Luís Silveira

Department of Mechanical Engineering, Centro de Tecnologia, EE/COPPE, Universidade Federal do Rio de Janeiro, Bloco G sala 204, Cidade Universitaria, CP 68503, CEP 21945-970 Rio de Janeiro, RJ, Brazil

Received 6 August 1999; in revised form 3 January 2000

Abstract

The shakedown analysis of structures under variable loads is considered, and the corresponding variational principles are presented. Separate formulations for elastic (or classical) shakedown, plastic shakedown and incremental collapse are compared. Two upper bounds for classical safety factors are presented and interpreted as the result of separate analysis concerning alternating plasticity and prevention of simple mechanisms of incremental collapse. As a first application, exact solutions are obtained for a closed tube under variable pressure and temperature. This example is the Bree problem with logarithmic temperature variation across the wall. A finite element procedure for shakedown analysis of tubes is then presented. This numerical procedure is validated by comparison with the exact solutions for the closed tube. Finally, the numerical method is applied to a restrained tube under variable pressure and logarithmic temperature. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Shakedown; Plasticity; Cyclic load; Variational formulations; Finite elements

1. Introduction

In this paper, the classical (elastic) shakedown approach is compared with safety analysis for prevention of incremental collapse and also with safety assessment against alternating plasticity (Maier, 1972, 1977; Christiansen, 1980; Fremond and Friaa, 1982; König, 1979, 1987; Polizzotto, 1993; Kamenjarzh, 1996; Mróz et al., 1995).

For a structure, made of a perfectly plastic material, under a combination of fixed and variable loads, the three types of failure modes are alternating plasticity (plastic shake down), incremental collapse (ratcheting), or instantaneous collapse (plastic collapse). Classical shakedown theory, here called elastic shakedown, deals with the prevention of any of the aforementioned phenomena. Indeed, the main objective in the analysis of structures under variable loads is the computation of the load amplification factor, μ, ensuring elastic shakedown.

* Corresponding author.

E-mail address: nestor@serv.com.ufrj.br (N. Zouain).

PII: S0020-7683(00)00165-7

The present article is organized in three parts. In the first part, Sections 2 and 3, variational principles for shakedown analysis are formulated and compared. These principles give the safety factor for elastic shakedown, μ, as the extremum value of static, kinematic or mixed constrained functionals. Similar principles are also stated for the safety factors ω and ρ concerning separate and exclusive prevention of alternating plasticity or simple mechanisms of incremental collapse (SMIC). The concept of SMIC is precisely defined there. Central to this discussion is the bounding relation $\mu \leqslant \min\{\omega, \rho\}$. The first part of this paper is the necessary background for the application given in the second part, and follows three previous works: Silveira and Zouain (1997), Zouain and Silveira (1999, 2000).

In the second part, Section 4, an application is considered. A variant of the classical tube problem, established by Bree (1967), as considered by Gokhfeld and Cherniavsky (1980), is solved and here in an exact form, almost closed. It consists of a closed thick tube under variable pressure and temperature with a logarithmic instantaneous pattern of temperature decay across the wall. Straightforward application of the kinematical principle for simple mechanisms of incremental collapse leads to the exact solution for the range of loading, where ratcheting is critical. The theoretical necessary condition, presented in the first part, is verified in this case, so as to ensure that no gap exists between the exact safety factor μ (elastic shakedown) and its upper bound ρ (SMIC).

In the third part of this paper, Section 5, we present a finite element procedure for shakedown analysis of tubes. This numerical procedure is validated by comparison with the exact solutions for the closed tube obtained in Section 4. Finally, the numerical method is applied to a restrained tube under variable pressure and logarithmic temperature. For the same problem, Hyde et al. (1985) performed step-by-step finite element analyses along specific cyclic loading histories which lead to stabilized ratchet strains. The shakedown/ratcheting boundary is predicted there by interpolation and extrapolation of the results of those incremental analyses. The present shakedown limit is the only direct solution, available in the literature, for the restrained tube considered, and it is in agreement with Hyde et al. results for specific cyclic loadings.

2. Shakedown analysis

Some basic notation is established first. Consider a body B and let $v \in V$ denote a velocity field (V is the functional space of admissible velocities). Likewise, $D \in W$ denotes a strain rate field, and $T \in W'$ is a stress field (W and W' are dual spaces). Let $\hat{T}$ and $\hat{D}$ represent values of the fields T and D at a point x of the body. Then, the internal power is denoted

$$\langle T, D\rangle = \int_{\mathrm{B}} \hat{T} \cdot \hat{D}\, \mathrm{dB}. \tag{1}$$

The kinematics relation is written as $D = \mathscr{D}v$, where $\mathscr{D}$ is the deformation operator. The set of self-equilibrated stress fields T^{r} (residual stresses) is denoted as S^{r} (an affine manifold, i.e. a set obtained by adding all the elements of a linear space to a fixed vector).

2.1. Material relationships

This paper is presented in the framework of elastic, ideally plastic materials, with associated plastic flow, characterized here by the set $P \subset W'$ of plastically admissible stress fields. The usual way of defining P is to choose an $\hat{m}$-vector valued function f describing the local yield criterion (Maier, 1972; Polizzotto, 1993). Then, the following local set is defined:

$$\hat{P} = \{\hat{T} \mid f(\hat{T}) \leqslant 0\}. \tag{2}$$

Likewise, the closed convex set P of plastically admissible stress fields is

$$T \in P \iff \{\hat{T} \in \hat{P} \quad \forall x \in \mathrm{B}\}. \tag{3}$$

The stress-free state of the body is assumed admissible, i.e. $T = 0 \in P$.

The plastic dissipation is defined by

$$\hat{\chi}(\hat{D}^{\mathrm{p}}) = \sup_{\hat{\mathrm{T}}^* \in \hat{\mathrm{P}}} (\hat{\mathrm{T}}^* \cdot \hat{\mathrm{D}}^{\mathrm{p}}) \tag{4}$$

with global counterpart: $\chi(D) = \sup_{T \in P} \langle T, D \rangle = \int_{\mathrm{B}} \hat{\chi}(\hat{D}) \mathrm{dB}$. These functions are sublinear, i.e. convex and positively homogeneous of the first degree.

The associated flow law can now be written in the compact form (Fremond and Friaa, 1982; Kamenjarzh, 1996; Borges et al., 1996)

$$T \in \partial\chi(D), \tag{5}$$

where $\partial\chi$ denotes the subdifferential of χ (Rockafellar, 1973). We briefly recall that the subdifferential of a function χ at a point D is the cone defined by the gradients of the function computed at all points adjacent to D at infinitesimal distance and in all directions. It coincides with the classical gradient in regular points. An equivalent definition is $T \in \partial\chi(D) \iff \{\chi(D^*) - \chi(D) \geqslant T \cdot (D^* - D) \quad \forall D^*\}$.

2.2. The domain of variable loading

The design data for shakedown analysis are a prescribed range of variable loading Δ° containing any feasible history of external loads, cyclic or not. External loading may include mechanical and thermal actions. Consequently, we better represent any external action, either a mechanical or a thermal load, by the stress field T^{e} which is the unique solution of the corresponding purely (or unlimited) elastic problem. Then, the data for shakedown analysis will be given in terms of a set Δ^{e} of (elastic) stress fields representing the domain of variation of mechanical and thermal loading.

Furthermore, shakedown principles can be stated using the envelope Δ of the domain of elastic stresses Δ^{e}, which is precisely defined in the following. For each point x of the body, consider the set $\hat{\Delta}$ of all stress tensors $\hat{T}$ which are local values of elastic stresses $T^{\mathrm{e}} \in \Delta^{\mathrm{e}}$ associated to any feasible loading. We now define the set Δ, in the space of stress fields, as constituted of all stress distributions T satisfying $\hat{T} \in \hat{\Delta}$ for any point x in the body.

As a mechanical interpretation, any virtual stress field T in the set Δ may be sought as collecting local values of elastic stresses, which are produced, at different instants, along a certain admissible loading program (cyclic or not).

Additionally, we call critical stresses program a particular stress field $T \in \Delta$ which is the solution of a shakedown problem. The set of thermo-mechanical external loadings which produce all the pointwise values of this stress field T constitute the critical loading program, or cycle, responsible for the incipient failure with lowest amplifying factor.

We give an example of a virtual stress field $T \in \Delta$ in the shakedown analysis of the closed tube under thermo-mechanical loading considered in Section 4. Indeed, the incremental collapse solution, $T \in \Delta$, representing the critical loading program for the closed tube of Section 4, complies with the following description. It coincides with the elastic stress produced by pure (peak) pressure, for any radial coordinate R smaller than a transition radius R_{t}. For the external part of the tube, i.e. $R \geqslant R_{\mathrm{t}}$, the active stress is the elastic stress produced by combined (peak) pressure and (peak) temperature. This completes the description of a stress field, suitable to represent the critical loading program, which is an element of the envelope of elastic stresses Δ defined above. Notice that this stress field cannot be associated, by elastic equations, to a single combination of pressure and temperature, i.e. $T \notin \Delta^{\mathrm{e}} \subset \Delta$ although $T \in \Delta$.

We define below a function that represents the maximum external power, over an assumed rate of deformation, obtained from any feasible loading history.

$$\hat{\varphi}(\hat{D}^{\mathrm{p}}) = \sup_{\hat{\mathrm{T}}^* \in \hat{\Delta}} (\hat{\mathrm{T}}^* \cdot \hat{\mathrm{D}}^{\mathrm{p}}). \tag{6}$$

The total counterpart is $\varphi(D^{\mathrm{p}}) = \sup_{T \in \Delta} \langle T, D \rangle = \int_{\mathrm{B}} \hat{\varphi}(\hat{D}^{\mathrm{p}}) \mathrm{dB}$. This is the support function of the set Δ, hence is sublinear.

2.3. Elastic shakedown

The main concern in shakedown analysis is to ensure elastic shakedown, thus preventing the three failure modes under variable loading, i.e. plastic shakedown, incremental collapse (or ratcheting) and plastic collapse (or "instantaneous" failure).

Bleich and Melan theorem ensures that any load factor μ^* is safe if there exists a fixed self-equilibrated stress field T^{r}, such that its superposition with any stress belonging to the amplified load domain $\mu^* \Delta$ is plastically admissible. The limit load factor μ for elastic shakedown is the supremum of safe factors; this leads to a variational principle.

Equilibrium formulation for elastic shakedown

$$\mu = \sup_{\substack{\mu^* \in R \\ T^{\mathrm{r}} \in W'}} \left\{ \mu^* \left| \begin{array}{l} \mu^* \Delta + T^{\mathrm{r}} \subset P \\ T^{\mathrm{r}} \in S^{\mathrm{r}} \end{array} \right. \right\}. \tag{7}$$

The notation used in the plastic admissibility constraint above means

$$\mu^* \Delta + T^{\mathrm{r}} \subset P \iff \{ \mu^* T + T^{\mathrm{r}} \in P \quad \forall T \in \Delta \}. \tag{8}$$

The solution of the equilibrium formulation for elastic shakedown consists of a translation T^{r} and an amplification factor μ. The translation T^{r} is constrained to belong to the affine manifold of residual stresses, S^{r}. The load domain Δ is translated and amplified, and the resulting set is constrained to remain entirely contained in the plastically admissible set P. The optimal T^{r} moves the prescribed set so as to allow for the maximal amplification, μ.

2.4. Plastic shakedown

Consider now the computation of the safety factor ω (for the design range of variations Δ) which prevents exclusively against plastic shakedown. Thus, it is only avoided here and all critical loading programs that produce some plastic deformation after any arbitrary large time, although the net strain increments vanish in some infinite sequence of instants. This mode of failure is called alternating plasticity because it frequently consists of equal direct and reverse plastic deformation at every cycle.

Following Polizzotto (1993), a load factor ω^* is safe with respect to plastic shakedown if there exists a fixed stress distribution T^o (not necessarily self-equilibrated) such that, when superposed to any stress belonging to the amplified domain of variation $\omega^* \Delta$, the yield criterion does not violate anywhere. This criterion leads to the following variational principle.

Equilibrium formulation for plastic shakedown

$$\omega = \sup_{\substack{\omega^* \in R \\ T^o \in W'}} \{ \omega^* \mid \omega^* \Delta + T^o \subset P \}. \tag{9}$$

Due to the definition of the sets P and Δ, the constraints in this variational principle restrains the values of the stress tensor at each point in the body independently of any other point. Consequently, computation of the limit load factor ω against plastic shakedown consists in solving uncoupled nonlinear problems for each point of the body and retaining the minimum value of the amplifying factors thus obtained.

The statical formulation for elastic shakedown, Eq. (7), may be obtained from the statical formulation for plastic shakedown Eq. (9), by just adding the self-equilibrium constraint. Hence, the modified supremum thus obtained, μ, can only be less or equal than the original, ω. This proves formally that ensuring elastic shakedown prevents plastic shakedown to occur, i.e. $\mu \leqslant \omega$. Several simple examples can be given, where this inequality is satisfied as an equality. Roughly speaking, this is the case when stress concentrations are critical, and so are alternating plasticity cycles.

Computing the safety factor ω against plastic shakedown as an estimation for the elastic shakedown safety factor μ is convenient because (i) it gives an upper bound, (ii) the estimation is exact in some range, and (iii) the computation of ω, by means of a sequence of local problems, is simpler than the computation of μ.

3. An upper bound

In this section, we look for an upper bound load factor ρ aiming the range of situations when ratcheting cycles are critical.

The upper bound proposal is based on the identifications of *simple mechanisms of incremental collapse*, brielfly SMIC, among general ratcheting mechanisms. An incremental collapse mechanisms, where plastic deformation accumulates in every cycle is called SMIC if in this incremental deformation field *no single point undergoes plastic strain more than once per cycle*.

It is proven in Zouain and Silveira (1999) that the prevention of simple mechanisms of incremental collapse, and only this, is assured by the safety factor ρ determined as follows.

Equilibrium formulation for SMIC

$$\rho = \inf_{T \in W'} \{\tilde{\rho}(T) \mid T \in \Delta\}, \tag{10}$$

where

$$\tilde{\rho}(T) = \sup_{\substack{\rho^* \in R \\ T^r \in W'}} \left\{ \rho^* \,\middle|\, \begin{array}{l} \rho^* T + T^r \in P \\ T^r \in S^r \end{array} \right\}. \tag{11}$$

Also, a change of variables gives

$$\tilde{\rho}(T) = \sup_{\substack{\rho^* \in R \\ T^c \in W'}} \{\rho^* \mid T^c \in P \cap S(\rho^* \mathscr{D}' T)\} \tag{12}$$

with $S(F)$ denoting the set of all stress fields equilibrated with the load F, and $\mathscr{D}'$, the equilibrium operator.

The optimization formulation that defines the functional $\tilde{\rho}(T)$ is a problem of limit analyses (see, for instance: Kamenjarzh, 1996; Fremond and Friaa, 1982; Christiansen and Andersen, 1998; and Borges et al., 1996). The stress field T determines the associated reference load $\mathscr{D}'T$, i.e. the data for limit analysis.

Statement (10) of the inadaptation analysis for simple mechanisms of incremental collapse is meaningful in physical terms. Indeed, this analysis may be interpreted as the problem of finding the stress field in the envelope of variable elastic stresses, minimizing the plastic collapse factor. We may take advantage of this concept in the development of numerical procedures aimed to compute the safety factor against SMIC. This has been treated in Silveira and Zouain (1997).

The inf–sup formulation in Eq.(10) is obtained in Zouain and Silveira (1999) as the dual problem of the statical principle (7) by interchanging the variables (T^r, μ) with T. This statical dualization, unlike the usual statical-kinematical dualization, defines a pair of dual problems which may effectively have different optimal values, thus presenting a duality gap.

A basic theorem of optimization theory (Rockafellar, 1973) ensures, under very general hypotheses, that the optimal value of an inf–sup problem is greater or equal than the optimal value of its sup–inf dual, i.e. the duality gap is non-negative. This is called the weak duality theorem in minimax theory. Any strong duality theorem, ensuring that the gap is zero, needs more restrictive hypotheses (Rockafellar, 1973; Christiansen, 1980). In the present context, weak duality proves the following result:

$$\mu \leqslant \rho, \tag{13}$$

i.e. ensuring elastic shakedown prevents simple mechanisms of incremental collapse to occur. Next, we briefly discuss on the usefulness of the present upper bound.

A simple plane stress situation with constrained deformation and thermo-mechanical loading is shown to present an effective gap between ρ and μ. It is the case of a small block, restrained in one direction, and made of a material that obeys the Mises criterion, when submitted to independent variable temperature and mechanical load. That is, the deformation in one direction of the plane of stresses is restrained, and a mechanical load acts in the orthogonal direction of this plane. The computations for this example are given in Zouain and Silveira (1999). This counterexample proves mathematically that no completely general theorem of strong duality may exist, for the formally dual problems representing static formulations of elastic shakedown and SMIC.

Nevertheless, an elementary bending problem under combined axial force and variable bending moment presents no gap between ρ and μ, as shown in Silveira and Zouain (1997). This demonstrates that we may frequently find the favorable situation $\mu = \rho$. Furthermore, the following proposition (Zouain and Silveira, 1999) proves that this case can be safely recognized.

Proposition 1. *A sufficient condition to ensure that there exists some simple mechanism of incremental collapse which is also critical for elastic shakedown analysis. i.e. that it holds*

$$\mu = \rho \tag{14}$$

is that the solutions ρ and T^{r} of the SMIC problem satisfy the following admissibility condition:

$$\rho \varDelta + T^{\mathrm{r}} \subset P. \tag{15}$$

In the sequel, the statical formulation for SMIC is transformed, by dualization, into a kinematical principle (Silveira and Zouain, 1997). Firstly, we write the limit analysis problem (11) in the following kinematical formulation:

$$\tilde{\rho}(T) = \inf_{v \in V} \{ \chi(\mathscr{D}v) \mid \langle T, \mathscr{D}v \rangle = 1 \}. \tag{16}$$

Then, substitution of Eq. (16) in Eq. (10), gives

Mixed formulation for SMIC

$$\rho = \inf_{\substack{v \in V \\ T \in W'}} \left\{ \chi(\mathscr{D}v) \,\middle|\, \begin{matrix} \langle T, \mathscr{D}v \rangle = 1 \\ T \in \varDelta \end{matrix} \right\}. \tag{17}$$

Furthermore, a main result in Silveira and Zouain (1997) is that the above variational problem also admits a purely kinematical formulation.

Kinematical formulation for SMIC

$$\rho = \inf_{v \in V} \{ \chi(\mathscr{D}v) \mid \varphi(\mathscr{D}v) = 1 \}. \tag{18}$$

Since χ and φ are positive homogeneous of the first degree, it also holds

$$\rho = \inf_{v\in V}\left\{\frac{\chi(\mathscr{D}v)}{\varphi(\mathscr{D}v)} \mid \varphi(\mathscr{D}v) > 0\right\}. \tag{19}$$

The stationary conditions of the SMIC analysis, corresponding to the extremum principles (10), (17) and (18), include the constraints $\langle T, \mathscr{D}v\rangle = 1$ and $T^{\mathrm{r}} \in S^{\mathrm{r}}$, together with

$$\rho T + T^{\mathrm{r}} \in \partial\chi(\mathscr{D}v), \tag{20}$$

$$T \in \partial\varphi(\mathscr{D}v). \tag{21}$$

These optimality conditions justify the name SMIC adopted to identify the particular class of failure mode that has been exclusively prevented in the bounding principles (10), (17) or (18).

In fact, we recognize above a purely plastic collapse mechanism v, whose strain rate field, $\mathscr{D}v$, is related, via the plastic flow (20), to a single critical stress field, $T^{\mathrm{c}} = \rho T + T^{\mathrm{r}}$. These critical stresses do not exist at every point simultaneously (at the same load); they occur sequentially along the critical cycle of loading. Likewise, the plastic strain rate field, $\mathscr{D}v$, is a cumulative mechanism of deformation, not necessarily a synchronous one. This type of velocity fields, v, are here called *simple mechanisms of incremental collapse* with respect to more complex mechanisms, also found among incremental collapse phenomena. These complex mechanisms produce compatible strain rates, $\mathscr{D}v$, which are *a sum of plastic deformations* at each point, not a single one. Complex mechanisms are realized, for instance, in the solution of the tube under thermo-mechanical loading considered in Zouain and Silveira (2000).

4. A closed tube under variable temperature and pressure

We present in this section analytical solutions for a closed thick tube submitted to independent variations of internal pressure and temperature. The instantaneous temperature pattern is logarithmic across the wall thickness and vanishes cyclically. The material behaves following von Mises model. This case, considered by Gokhfeld and Cherniavsky (1980, p. 167–175), is a variant of the fundamental Bree problem (Bree, 1967; Ponter and Karadeniz, 1985; Robinson, 1991; Phan, 1995).

Consider a long tube with closed ends. The internal and external radii are R_{int} and R_{ext}, respectively. The radial coordinate R is substituted by the dimensionless radius r, given below, together with the relevant geometric parameter ℓ.

$$r := \frac{R}{R_{\mathrm{ext}}}, \qquad \ell := \frac{R_{\mathrm{ext}}}{R_{\mathrm{int}}}. \tag{22}$$

4.1. Loading conditions

The internal pressure p_{int} varies between 0 and $\bar{p}_{\mathrm{int}}$. Accordingly, the dimensionless mechanical parameter is defined as

$$p := \frac{p_{\mathrm{int}}}{(\ell^2 - 1)\sigma_{\mathrm{Y}}}, \tag{23}$$

where σ_{Y} denotes the yield stress. Then, p varies between 0 and $\bar{p} := \bar{p}_{\mathrm{int}}/(\ell^2 - 1)\sigma_{\mathrm{Y}}$.

Plastic collapse of the tube is produced at the following internal pressure:

$$p_{\mathrm{c}} = \frac{2}{\sqrt{3}}\sigma_{\mathrm{Y}} \ln \ell. \tag{24}$$

This suggests the use of an additional dimensionless parameter, defined as

$$\hat{p} := \frac{p_{\rm int}}{p_{\rm c}} = \sqrt{3}\beta p \tag{25}$$

varying between 0 and $\underline{\hat{p}} := \sqrt{3}\beta\bar{p}$. We used above, for convenience, the expression

$$\beta := \frac{\ell^2 - 1}{2\ln\ell}. \tag{26}$$

Notice that $\ell < \beta < \ell^2$ because $\ell > 1$. Further, the approximations representing thin tubes are obtained for $\ell \to 1^+$, that implies $\beta \to 1^+$.

Independently from pressure, the difference between internal and external wall temperatures, $\Theta_{\rm int} - \Theta_{\rm ext}$, varies between 0 and $\bar{\Theta}$. The temperature at a distance r of the axis is assumed to follow, at any instant, the steady state pattern

$$\Theta = \Theta_{\rm ext} - (\Theta_{\rm int} - \Theta_{\rm ext})\frac{\ln r}{\ln\ell}. \tag{27}$$

Then, a suitable dimensionless thermal parameter is

$$q := \frac{Ec_\Theta(\Theta_{\rm int} - \Theta_{\rm ext})}{2\sigma_{\rm Y}(1-v)(\ell^2-1)}, \tag{28}$$

where E denotes Young's modulus, v is Poisson's coefficient, and c_Θ is the thermal expansion coefficient. Consequently, the prescribed limits for temperature loading are 0 and $\bar{q} := Ec_\Theta\bar{\Theta}/2\sigma_{\rm Y}(1-v)(\ell^2-1)$, in dimensionless form.

In order to produce Bree-type diagrams in the usual standards, we define the additional dimensionless thermal parameter

$$\hat{q} := \frac{Ec_\Theta(\Theta_{\rm int} - \Theta_{\rm ext})}{2\sigma_{\rm Y}(1-v)} = (\ell^2-1)q \tag{29}$$

with bounds 0 and $\underline{\hat{q}} := (\ell^2-1)\bar{q} = {\rm E}c_\Theta\bar{\Theta}/2\sigma_{\rm Y}(1-v)$.

4.2. Elastic stresses

External loading for the tube is given, in shakedown analysis, by the elastic stress solutions: T^p, under pure pressure, and T^q, under pure thermal loading. These stress fields are given below, in dimensionless form, by using the reduced stress tensors $\tilde{T} := (1/\sigma_{\rm Y})T, \tilde{T}^p := (1/p\sigma_{\rm Y})T^p$, and $\tilde{T}^q := (1/q\sigma_{\rm Y})T^q$. Accordingly, variable loading produce the following elastic stress:

$$\widetilde{T} = p\widetilde{T}^p + q\widetilde{T}^q, \tag{30}$$

where the basic elastic fields are

(i) *Elastic stresses due to pressure loading*

$$\widetilde{T}^p_r = 1 - r^{-2}, \qquad \widetilde{T}^p_\theta = 1 + r^{-2}, \qquad \widetilde{T}^p_z = 1. \tag{31}$$

Hence, the deviatoric components are $\widetilde{S}^p_r = -r^{-2}, \widetilde{S}^p_\theta = r^{-2}$, and $\widetilde{S}^p_z = 0$.

(ii) *Elastic stresses due to temperature loading*

$$\widetilde{T}^q_r = r^{-2} - 1 + 2\beta\ln r, \tag{32}$$

$$\widetilde{T}^q_\theta = -r^{-2} - 1 + 2\beta(1 + \ln r), \tag{33}$$

$$\widetilde{T}_z^q = 2[-1 + \beta(1 + 2\ln r)]. \tag{34}$$

Therefore, the mean stress is $\widetilde{T}_m^q = \frac{4}{3}[-1 + \beta(1 + 2\ln r)]$, and the deviatoric components are

$$\tilde{S}_r^q = \tfrac{1}{3}[1 + 3r^{-2} - 2\beta(2 + \ln r)], \tag{35}$$

$$\tilde{S}_\theta^q = \tfrac{1}{3}[1 - 3r^{-2} + 2\beta(1 - \ln r)], \tag{36}$$

$$\tilde{S}_z^q = \tfrac{2}{3}[-1 + \beta(1 + 2\ln r)]. \tag{37}$$

The local domain of variable loading, $\Delta(r)$, is a parallelogram with four vertices $\{\tilde{T}^k(r);\ k = 1, \ldots, 4\}$ given by Eq. (30) with $(p, q) = \{(0,0), (0,\bar{q}), (\bar{p},\bar{q}), (\bar{p},0)\}$.

4.3. Plastic shakedown

Firstly, it is recognized that the critical points, where the largest variations of elastic stresses take place are all the points on the internal surface of the cylinder. Consequently, the analysis reduces to a local optimization problem associated to the domain of elastic stress variation $\Delta(r)$ for $r = \ell^{-1}$.

Secondly, it is also recognized that a critical alternating plasticity cycle may consist of yielding under pure pressure followed by reverse yielding under pure temperature loading. Accordingly, the solution of the local optimization problem translates and amplifies the quadrilateral $\Delta(\ell^{-1})$, so that the two opposite vertices $\bar{p}\sigma_Y\widetilde{T}^p$ and $\bar{q}\sigma_Y\widetilde{T}^q$ become a diameter of the Mises cylinder, whose radius is $\sqrt{2/3}\sigma_Y$.

Thus, the critical amplified parameters $(p, q) = (\omega\bar{p}, \omega\bar{q})$ satisfy the following condition:

$$\|\omega\bar{q}\tilde{S}^q(\ell^{-1}) - \omega\bar{p}\tilde{S}^p(\ell^{-1})\| = 2\sqrt{\tfrac{2}{3}}. \tag{38}$$

Substitution of Eq. (31), (35), (36), and (37) in the above condition leads to

$$3(\omega\bar{p})^2\ell^4 + 4(\omega\bar{q})^2(\ell^2 - \beta)^2 + 6(\omega\bar{p})(\omega\bar{q})\ell^2(\ell^2 - \beta) = 4, \tag{39}$$

which is an ellipse in a diagram for the critical values $(p, q) = (\omega\bar{p}, \omega\bar{q})$. Moreover, we can write the expression for the alternating plasticity safety factor, for prescribed bounds $(\bar{p}, \bar{q})$, as

$$\omega = \frac{2}{\sqrt{3\bar{p}^2\ell^4 + 4\bar{q}^2(\ell^2 - \beta)^2 + 6\bar{p}\bar{q}\ell^2(\ell^2 - \beta)}}. \tag{40}$$

The limit for alternating plasticity, i.e. ellipse (39), is written below in terms of the critical parameters $(\hat{p}, \hat{q}) = (\omega\underline{\hat{p}}, \omega\underline{\hat{q}})$ (Eqs. (25) and (29)), and plotted in the Bree-type diagram of Fig. 1.

$$(\omega\underline{\hat{p}})^2\ell^4(\ell^2 - 1)^2 + 4(\omega\underline{\hat{q}})^2\beta^2(\ell^2 - \beta)^2 + 2\sqrt{3}(\omega\underline{\hat{p}})(\omega\underline{\hat{q}})\beta\ell^2(\ell^2 - 1)(\ell^2 - \beta) = 4\beta^2(\ell^2 - 1)^2. \tag{41}$$

For moderately thin tubes, and in the range of application of this limit, shown in Fig. 1, this ellipse is very close to its tangent at $(\omega\underline{\hat{p}}, \omega\underline{\hat{q}}) = (0, \ell^2 - 1/\ell^2 - \beta)$

$$\frac{\sqrt{3}\ell^2(\ell^2 - 1)}{4\beta(\ell^2 - \beta)}\omega\underline{\hat{p}} + \omega\underline{\hat{q}} = \frac{\ell^2 - 1}{\ell^2 - \beta}. \tag{42}$$

Further, the curve corresponding to alternating plasticity of thin tubes is obtained from Eq. (41) or Eq. (42) with $\ell \to 1^+$, and reads

$$\frac{\sqrt{3}}{2}\omega\underline{\hat{p}} + \omega\underline{\hat{q}} = 2. \tag{43}$$

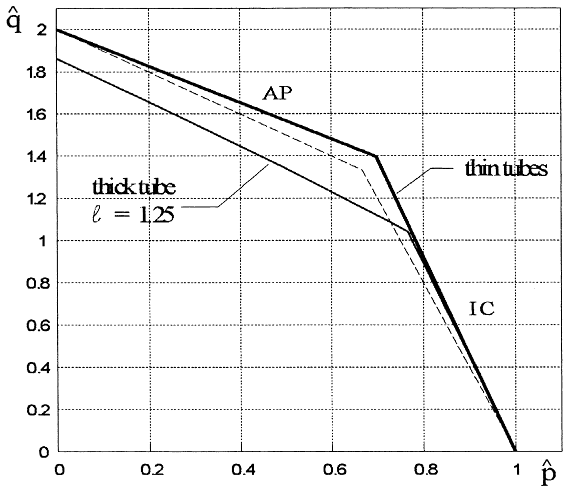

Fig. 1. Shakedown boundaries for closed thick tubes under independent variations of pressure and (logarithmic) temperature. Bree-type diagram: thermal load $\hat{q}$, Eq. (29), versus pressure $\hat{p}$, Eq. (25). AP: alternating plasticity, IC: incremental collapse, (- - -): extended Bree solution.

This exact theoretical solution for thin tubes is close to the approximation given by the extended Bree solution $\hat{p} + \hat{q} = 2$ (see, for instance, Robinson, 1991). Both the lines are shown in Fig. 1.

Finally, we recall that alternating plasticity becomes the critical regime under pure pressure loadings for very thick tubes, those with $\ell \geqslant 2.2185$, as pointed out by Lubliner (1990, p. 213). This result can be derived from Eq. (41) by setting $\underline{\hat{q}} = 0$, thus obtaining $\omega\underline{\hat{p}} = 2\beta\ell^{-2}$, and then comparing this critical pressure with the instantaneous collapse pressure $\hat{p} = 1$. Hence, the transition cylinder verifies $2\beta\ell^{-2} = 1$, which, in view of the definition (26) of β, leads to the equation $1 - \ell^{-2} = \ln \ell$, with the solution $\ell = 2.2185$.

4.4. Incremental collapse

In order to compute the safety factor ρ for simple mechanisms of incremental collapse, by means of the kinematical formulation, we consider general expressions for plastic mechanisms. Let $v_r = V_r/R_{\text{ext}}$ and $v_z = V_z/R_{\text{ext}}$ denote dimensionless counterparts of the radial and axial velocities V_r and V_z. For a long tube, the deformation rates are (Lubliner, 1990)

$$D_r = \frac{\mathrm{d}v_r}{\mathrm{d}r}, \qquad D_\theta = \frac{v_r}{r}, \qquad D_z = \text{constant}. \tag{44}$$

It follows from $\operatorname{tr} D = 0$ that $\mathrm{d}(rv_r)/\mathrm{d}r + D_z r = 0$, and hence the velocity field depends on two constants, c_v and D_z, such that

$$v_r = \frac{c_v}{r} - \frac{1}{2} D_z r. \tag{45}$$

Then

$$D_r = -c_v r^{-2} - \tfrac{1}{2} D_z, \qquad D_\theta = c_v r^{-2} - \tfrac{1}{2} D_z. \tag{46}$$

and $\|D\| = \sqrt{4c_v^2 r^{-4} + 3D_z^2}/\sqrt{2}$.

As a consequence, the optimization problem, given by the kinematical formulation of SMIC, Eq. (19), becomes a mathematical programming problem in the variables c_v and D_z. The objective function is the quotient of the plastic dissipation $\chi(D)$ over the maximum power of loads $\varphi(D)$. No constraints are present. Moreover, this kinematical principle reduces to a one-variable problem, because it can be chosen, for instance, $c_v = 1$, due to the fact that $\chi(D)$ and $\varphi(D)$ are homogeneous of the first degree.

The solution to the kinematical formulation (19) is developed in the sequel.

4.4.1. Plastic dissipation

The total and local plastic dissipation functions are

$$\chi(D) = 2\pi R_{\text{ext}}^2 \int_{\ell^{-1}}^{1} \hat{\chi}(D(r)) r \, dr, \tag{47}$$

$$\hat{\chi}(D(r)) = \sqrt{\frac{2}{3}} \sigma_{\text{Y}} \| D(r) \|. \tag{48}$$

Then, substituting Eq. (46)

$$\chi(D) = \frac{2\pi}{\sqrt{3}} R_{\text{ext}}^2 \sigma_{\text{Y}} \int_{\ell^{-1}}^{1} \sqrt{4c_v^2 r^{-4} + 3D_z^2} \, r \, dr. \tag{49}$$

By performing the integration above, the following exact expression of the plastic dissipation for a general mechanism is obtained.

$$\chi(D) = \frac{\pi}{\sqrt{3}} R_{\text{ext}}^2 \sigma_{\text{Y}} \left(4c_v \ln \ell + \sqrt{4c_v^2 + 3D_z^2} - \sqrt{4c_v^2 + 3D_z^2 \ell^{-4}} - 2c_v \ln \frac{2c_v + \sqrt{4c_v^2 + 3D_z^2}}{2c_v + \sqrt{4c_v^2 + 3D_z^2 \ell^{-4}}} \right). \tag{50}$$

4.4.2. Maximum power of loads

Recalling Eq. (6), the total and local maximum power functions for loading are

$$\varphi(D) = 2\pi R_{\text{ext}}^2 \int_{\ell^{-1}}^{1} \hat{\varphi}(D(r)) r \, dr, \tag{51}$$

$$\hat{\varphi}(D(r)) = \sup_{k=1,\ldots,4} T^k(r) \cdot D(r), \tag{52}$$

where, due to Eq. (46)

$$T^k(r) \cdot D(r) = c_v r^{-2} [T_\theta^k(r) - T_r^k(r)] + \tfrac{3}{2} D_z S_z^k(r). \tag{53}$$

With respect to the supremum in Eq. (52), we assume now, and confirm a posteriori, that (i) the load system 4, i.e. $T^4 = \bar{p}\sigma_{\text{Y}}\widetilde{T}^p$, is active for $r \in (\ell^{-1}, r_{\text{t}})$, and (ii) the load system 3, i.e. $T^3 = \bar{p}\sigma_{\text{Y}}\widetilde{T}^p + \overline{q}\sigma_{\text{Y}}\widetilde{T}^q$, is active for $r \in (r_{\text{t}}, 1)$. The transition radius r_{t} is obtained from the following condition:

$$T^3(r) \cdot D(r) = T^4(r) \cdot D(r). \tag{54}$$

But $T^3 = \overline{q}\sigma_{\text{Y}}\widetilde{T}^q + T^4$; hence $\widetilde{T}^q(r) \cdot D(r) = 0$ and then

$$2c_v r_{\text{t}}^{-2}(r_{\text{t}}^{-2} - \beta) + D_z[1 - \beta(1 + 2\ln r_{\text{t}})] = 0. \tag{55}$$

Hence, the ratio D_z/c_v is the function of r_{t} given by

$$\frac{D_z}{c_v} = \frac{2(\beta r_{\text{t}}^2 - 1)}{r_{\text{t}}^4[1 - \beta(1 + 2\ln r_{\text{t}})]}. \tag{56}$$

From now on, we consider the kinematical formulation at hand as an optimization problem in terms of r_{t} while c_v and D_z are dependent variables given by Eq. (56).

Recalling the previous hypothesis, concerning active loading, it can be written

$$\int_{\ell^{-1}}^{1} \hat{\varphi}(D(r)) r \, \mathrm{d}r = \int_{\ell^{-1}}^{r_t} T^4 \cdot Dr \, \mathrm{d}r + \int_{r_t}^{1} T^3 \cdot Dr \, \mathrm{d}r \tag{57}$$

$$\int_{\ell^{-1}}^{1} \hat{\varphi}(D(r)) r \, \mathrm{d}r = \rho \bar{p} \sigma_Y \int_{\ell^{-1}}^{1} \widetilde{T}^{\mathrm{p}} \cdot Dr \, \mathrm{d}r + \rho \bar{q} \sigma_Y \int_{r_t}^{1} \widetilde{T}^{q} \cdot Dr \, \mathrm{d}r. \tag{58}$$

Now, using Eqs. (53) and (31)

$$\int_{\ell^{-1}}^{1} \widetilde{T}^{p} \cdot Dr \, \mathrm{d}r = c_v(\ell^2 - 1). \tag{59}$$

Similarly, from Eqs. (53), (35), (36) and (37)

$$\int_{r_t}^{1} \widetilde{T}^{q} \cdot Dr \, \mathrm{d}r = \int_{r_t}^{1} \{2c_v(\beta r^{-1} - r^{-3}) + D_z[(\beta - 1)r + 2\beta r \ln r]\} \mathrm{d}r \tag{60}$$

$$\int_{r_t}^{1} \widetilde{T}^{q} \cdot Dr \, \mathrm{d}r = c_v(1 - r_t^{-2} - 2\beta \ln r_t) + \frac{1}{2} D_z[-1 + r_t^2(1 - 2\beta \ln r_t)]. \tag{61}$$

Finally, introducing Eqs. (59), (61), and (58) in Eqs. (51), we arrive to the expression of the maximum power of loading

$$\varphi(D) = \pi R_{\mathrm{ext}}^2 \sigma_Y \{2c_v[\rho \bar{p}(\ell^2 - 1) + \rho \bar{q}(1 - r_t^{-2} - 2\beta \ln r_t)] + D_z \rho \bar{q}[-1 + r_t^2(1 - 2\beta \ln r_t)]\}. \tag{62}$$

4.4.3. The boundary for incremental collapse

When numerical values for the limits $(\bar{p}, \bar{q})$ are prescribed, the set of relations (50), (62), and (56) completely determine the one-variable minimization problem whose optimal value is the safety factor, ρ, for simple mechanisms of incremental collapse

$$\rho = \inf_{r_t} \frac{\chi(D)}{\varphi(D)}. \tag{63}$$

To this purpose, an iterative procedure is required. For instance, the Newton method may be used. The resulting curve of critical pairs $(\rho\hat{p}, \rho\underline{\hat{p}})$, for a tube with $\ell = 1.25$ is plotted in Fig. 1.

Additionally, we verified in this application the necessary condition (15) of Proposition 1 to guarantee that the computed amplifying factor, ρ, restricted to simple mechanisms is in fact the true solution, μ, of the elastic shakedown analysis. Verifying this necessary condition requires the computation of the residual stress field, T^{r}, and afterwards proving that $T^{\mathrm{r}} + T^k$ is plastically admissible for $k = 1, \ldots, 4$ and for all r. Since we have now the local value for the strain rate, $D(r)$, the constitutive relation gives the corresponding stress, say $T^{\mathrm{c}}(r)$. This stress $T^{\mathrm{c}}(r)$ equals $T^{\mathrm{r}}(r) + \rho T^4(r)$ in $r \in (\ell^{-1}, r_t)$, and equals $T^{\mathrm{r}}(r) + \rho T^3(r)$ in $r \in (r_t, 1)$. Thus, we compute the residual stress by using $T^{\mathrm{r}}(r) = T^{\mathrm{c}}(r) - \rho T^4(r)$ in $r \in (\ell^{-1}, r_t)$, and $T^{\mathrm{r}}(r) = T^{\mathrm{c}}(r) - \rho T^3(r)$ in $r \in (\ell^{-1}, 1)$.

In summary, the necessary condition, (Eq. (15)) to prove that the solution of the SMIC problem is the exact safety factor of elastic shakedown, i.e. that

$$\mu = \rho \tag{64}$$

is verified for the tube under thermo-mechanical loading considered.

Although the exact solution of this shakedown analysis is now complete, we give in the sequel a simple upper bound, which happens to be very close to the exact value. This upper bound is obtained by choosing

$$r_t^{\mathrm{ub}} = \beta^{-1/2}, \tag{65}$$

and then computing the corresponding fraction $\chi(D)/\varphi(D)$. Notice that for the above value of r_t, Eq. (56) gives $D_z = 0$. Then, from Eqs. (50) and (62), it follows that

$$\mu \leqslant \rho^{ub} := \frac{2\ln\ell}{\sqrt{3}\{\bar{p}(\ell^2-1)+\bar{q}[1-\beta(1-\ln\beta)]\}}. \tag{66}$$

This gives the following straight line in the Bree-type diagram of Fig. 1:

$$\rho^{ub}\underline{\hat{p}} + \frac{\sqrt{3}\beta[1-\beta(1-\ln\beta)]}{(\ell^2-1)^2}\rho^{ub}\underline{\hat{q}} = 1. \tag{67}$$

Notice that $\rho^{ub}\underline{\hat{p}} = 1$ for $\underline{\hat{q}} = 0$.

For a tube with $\ell = 1.25$, shown in Fig. 1, and with prescribed loading limits $(\underline{\hat{p}}, \underline{\hat{q}}) = (0.800716,\ 0.895410)$, the exact solution is $\mu = 1.000000$, with $r_t = 0.891375$ and $D_z/C_v = 0.155800$. In this case, the upper bound is $\rho^{ub} = 1.005992$ (0.6% error), for $r_t^{ub} = 0.890730$.

The curve corresponding to critical combinations $(\rho\underline{\hat{p}}, \rho\underline{\hat{q}})$ for incremental collapse of thin tubes is obtained by using the same minimization procedure and with $\ell \to 1^+$. However, a simpler way to obtain the same result is to take limits in Eq. (67). This leads to

$$\rho\underline{\hat{p}} + \frac{\sqrt{3}}{8}\rho\underline{\hat{q}} = 1. \tag{68}$$

Likewise in alternating plasticity, this exact theoretical solution for thin tubes, shown in Fig. 1, is close to the approximation given by the extended Bree solution $\hat{p} + 0.25\hat{q} = 1$ (Robinson, 1991).

Finally, for thin tubes, the intersection of Eqs. (68) and (43) gives the set of parameters $\underline{\hat{q}} = 2\underline{\hat{p}} = 8(4-\sqrt{3})/13$ producing $\mu = \omega = \rho = 1$, i.e. the boundary between the regime of pure alternating plasticity and ratcheting.

5. Finite element solutions for tubes

In this section, a mixed finite element of shakedown analysis of tubes is described. The resulting numerical procedure is validated by comparing with the exact solution for the closed tube, given in the previous section, and then applied to the case of a restrained tube under pressure and logarithmic temperature variations.

The formulation of the finite element procedure is briefly considered here. However, the mathematical programming algorithm used for solving the discrete optimization problem, which was specially developed for shakedown analysis, is reported elsewhere.

The one-dimensional finite element discretization presented in the sequel is suited for long thick tubes under internal and external pressure. The basic temperature patterns, across the wall thickness, associated to each variable thermal parameter are arbitrary. The end conditions of the tubes may be fixed ends or those corresponding to open or closed tubes, as described in Lubliner (1990).

Avoiding the FE locking phenomenon is important in the considered solids, with symmetry of revolution in the geometry and loadings, and undergoing purely plastic flow. Locking is prevented here by using a mixed interpolation. Accordingly, the weak conditions of equilibrium and zero velocity divergence are taken into account to select the stress and velocity shape functions.

The radial component of the velocity v_r is interpolated quadratic in each finite element and continuous across element. The axial component of the strain rate D_z is constant inside the element. The mean stress T_m is also constant in each element while all components of the deviatoric stress S are linear in any element and discontinuous at interelement nodes. The generic element i has three nodes: the left and right ones are 1 and 2, respectively and the mid-point is 3. The vectors of nodal parameters for the ith element, v^i for velocities and T^{ri} for residual stress components, are

$$v^i = [v_r^1 \quad v_r^2 \quad v_r^3 \quad D_z^3]^{\mathrm{T}}, \tag{69}$$

$$T^{ri} = [S_r^{r1} \quad S_\theta^{r1} \quad S_z^{r1} \quad S_r^{r2} \quad S_\theta^{r2} \quad S_z^{r2} \quad T_m^3]^{\mathrm{T}}, \tag{70}$$

where the superscript T indicates matrix transposition.

The strain rate field $D(r)$ and the residual stress distribution $T^{\mathrm{r}}(r)$ are obtained by means of the finite element interpolations

$$D(r) = \mathscr{D}N_v^i(r)v^i, \qquad T^{\mathrm{r}}(r) = N_T^i(r)T^{\mathrm{r}i}, \tag{71}$$

where $N_v^i(r)$ and $N_T^i(r)$ are the interpolation matrices for velocity and stress. This leads to the following discrete form of self-equilibrium written in terms of the global vector of residual stress parameters $\underline{T}^{\mathrm{r}}$ (collecting T^{ri} for all elements)

$$B^T \underline{T}^{\mathrm{r}} = 0, \tag{72}$$

where the matrix B is assembled from its element counterpart

$$B^i = 2\pi R_{\mathrm{ext}}^2 \int_{r_1}^{r_2} N_{\mathrm{T}}^{i\mathrm{T}}(r)\mathscr{D}N_v^i(r)r\,\mathrm{d}r \tag{73}$$

with r_1 and r_2 denoting the non-dimensional radial coordinates of the left and right nodes of element i.

Consider now the basic elastic stress profiles defining the domain of loading variations. Let $T^k(r)$, with $k = 1, \dots, n_\ell$, be the prescribed elastic stress associated to each vertex of the load domain in the finite-dimensional space of loading parameters. For instance, in the present applications, concerning independent variations of the pressure parameter $p \in (0, \bar{p})$ and the thermal parameter $q \in (0, \bar{q})$, there are $n_\ell = 4$ vertices, identified by $(p, q) = \{(0,0), (0,\bar{q}), (\bar{p},\bar{q}), (\bar{p},0)\}$ and $T^1(r) = 0, T^2(r) = \bar{q}T^q(r), T^3(r) = \bar{p}T^p(r) + \bar{q}T^q(r)$, and $T^4(r) = \bar{p}T^p(r)$, with $T^p(r)$ and $T^q(r)$ denoting the stress distributions due to single unit pressure or thermal loadings (given in Eqs. (30)–(34) for the closed tube, and in Eqs. (79)–(82) for the restrained tube).

The element vectors collecting nodal values of the prescribed elastic stresses are chosen in accordance with definition (70) of the vector of nodal stress parameters, i.e.

$$T^{ki} = \left[S_r^{k1} \quad S_\theta^{k1} \quad S_z^{k1} \quad S_r^{k2} \quad S_\theta^{k2} \quad S_z^{k2} \quad T_m^{k3}\right]^{\mathrm{T}} \tag{74}$$

for $k = 1, \dots, n_\ell$ and $i = 1, \dots, n_e$, with n_ℓ denoting the number of basic loads (vertices) and n_e denoting the number of elements. In the above equation, S_r^{k1}, S_θ^{k1}, S_z^{k1} are the deviatoric components of the elastic stress $T^k(r)$ at the left node of element i. The above symbol T_m^{k3}, representing mean stress all along the element, is defined as the arithmetic mean of the mean stress components of the elastic stress tensor $T^k(r)$ computed at both ends of the element.

The plastic admissibility constraint in Bleich–Melan's principle (7), in terms of the combined stress fields $\mu\widetilde{T}^k(r) + T^{\mathrm{r}}(r)$, is now imposed at the left and right nodes of each element. This is accomplished by simply restraining the nodal values. For the von Mises model, the following two constraints must be imposed for each element i:

$$f_1(T^i) := \tfrac{3}{2}\left[(S_r^1)^2 + (S_\theta^1)^2 + (S_z^1)^2\right] - \sigma_{\mathrm{Y}}^2 \leqslant 0, \tag{75}$$

$$f_2(T^i) := \tfrac{3}{2}\left[(S_r^2)^2 + (S_\theta^2)^2 + (S_z^2)^2\right] - \sigma_{\mathrm{Y}}^2 \leqslant 0, \tag{76}$$

where

$$T^i := \mu T^{ki} + T^{\mathrm{r}i}. \tag{77}$$

The interpolations above must now be introduced in a mixed extremum principle for elastic shakedown as the mixed formulation given in Zouain and Silveira (2000). The resulting discrete optimization problem may then be put in one of its dual forms. For instance, the discrete equilibrium formulation for elastic shakedown is

$$\mu = \sup_{\mu^*, \underline{T}^{\mathrm{r}}} \mu^* \left| \begin{array}{l} B^T \underline{T}^{\mathrm{r}} = 0 \\ f_1(\mu T^{ki} + T^{\mathrm{r}i}) \leqslant 0 \\ f_2(\mu T^{ki} + T^{\mathrm{r}i}) \leqslant 0, \quad k = 1, \ldots, n_\ell, \quad i = 1, \ldots, n_e. \end{array} \right. \tag{78}$$

This discrete optimization problem is solved here by using an algorithm based on a Newton-like iteration for the equalities pertaining to the set of optimality conditions. A correction is then performed in order to preserve plastic admissibility in each iteration.

The present finite element model for shakedown analysis of tubes was applied to the case of the closed-end tube, described in the previous section, in order to validate the numerical procedure by comparing with the exact solution already obtained. In this case, the finite element results reproduce the exact solutions of the thick tube in Fig. 1, to within an error of 0.4%, when 20 finite elements are used.

5.1. A restrained tube under variable temperature and pressure

In this section, we present numerical solutions for a fixed-end thick tube submitted to independent variations of internal pressure and temperature. The material and the loading conditions are the same considered in previous sections for the case of a closed tube. That is, at any instant, the temperature decay through the wall thickness is logarithmic.

Due to the axial restraint, the simple mechanisms of incremental collapse, that solved exactly the closed-end tube, are no longer critical, as detected in the finite element solution.

The notation here is as defined in Section 4.

The four basic loadings $T^k(r)$ are determined from the following stress fields:

(i) *Elastic stresses due to pressure loading*

$$\tilde{T}_r^p = 1 - r^{-2}, \qquad \tilde{T}_\theta^p = 1 + r^{-2}, \qquad \tilde{T}_z^p = 2\nu. \tag{79}$$

(ii) *Elastic stresses due to temperature loading*

$$\tilde{T}_r^q = r^{-2} - 1 + 2\beta \ln r, \tag{80}$$

$$\tilde{T}_\theta^q = -r^{-2} - 1 + 2\beta(1 + \ln r), \tag{81}$$

$$\tilde{T}_z^q = 2[\nu(\beta - 1) + 2\beta \ln r]. \tag{82}$$

The finite element procedure for shakedown analysis is applied to thin tubes ($\ell \leqslant 1.1$) and a thick tube with $\ell = 1.25$. The results are shown in Fig. 2.

The same example of application is treated in Hyde et al. (1985) by performing incremental finite element analysis under assumed loading cycles. Then, the stabilized ratchet cycle is identified so as to measure the ratchet strain. Some contours of constant ratchet strain are produced by interpolation or extrapolation over a set of results (a family of curves is fitted to the data). Finally, the shakedown/ratcheting boundary is predicted, in Hyde et al., by extrapolation of these values.

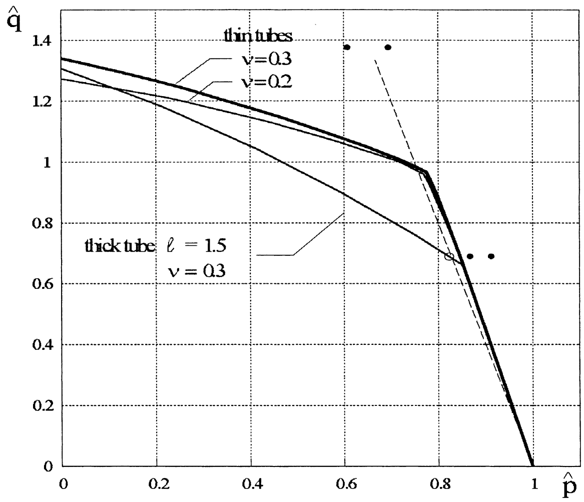

Fig. 2. Shakedown boundaries for fixed-end thick tubes under independent variations of pressure and (logarithmic) temperature. Bree-type diagram: thermal load $\hat{q}$, Eq. (29), versus pressure $\hat{p}$, Eq. (25). (- - -): extended Bree solution. (•): specific cycles producing incremental collapse, obtained by Hyde et al. (1985). (∘): extrapolated ratcheting boundary for thin tubes, predicted by Hyde et al. (1985).

Present results are compared, in Fig. 2, with those obtained by Hyde et al. We observe that

1. The points $(\hat{p},\hat{q}) = (0.8660, 0.688)$ and $(\hat{p},\hat{q}) = (0.9099, 0.688)$, marked in Fig. 2, correspond to effective incremental collapse, with steady state ratchet strains equal to 0.111 and 0.363, respectively, as reported in Table 4 of Hyde et al. (1985). Both points belong to the incremental collapse domain of the present solution and they are very close to our shakedown/ratcheting boundary.
2. The point $(\hat{p},\hat{q}) = (0.82, 0.688)$ predicted by Hyde et al. to be at the zero ratchet strain contour, by means of extrapolation, is also very close to our direct solution.

Therefore, the present numerical results are in good agreement with those of Hyde et al. (1985).

6. Conclusions

In the first part of the paper, we discuss the upper bound $\mu_{\text{ub}} := \min\{\omega, \rho\}$ for the classical shakedown safety factor μ, with ω associated to alternating plasticity and ρ to simple mechanisms of incremental collapse. This upper bound is simpler for analytical or numerical computations and admits a meaningful interpretation. The generation of the corresponding variational principles is based on dualization techniques of convex analysis. A brief summary of previous results and applications of this theoretical background is also given here. Finite element approximations using this bounding formulation can be found in Silveira and Zouain (1997). However, the focus in the present work is on the exact analytical solution of the Bree problem, given in Section 4.

A closed thick tube under independent variations of internal pressure and wall temperature gradient, assumed logarithmic, is considered in Section 4. Once the model assumptions are established, no further approximations are present in solution, which is then called exact. For the range of loadings that produce ratcheting, the exact solution is obtained by solving the bounding kinematical formulation. Most of all, Proposition 1 is then recalled to ensure that the computed solution, restricted to simple mechanisms of incremental collapse, is in fact the true solution in the elastic shakedown analysis.

This example of application, in Section 4, emphasizes the crucial role of the necessary condition, given by Eq. (15) in Proposition 1 to identify whenever the upper bound coincides with the exact elastic shakedown factor.

Several linear approximations, for thick and thin closed tubes, are also presented and discussed in Section 4. The shakedown diagram for the closed tubes, given in Fig. 1, compares well with existing approximations (Bree, 1967; Gokhfeld and Cherniavsky, 1980; Ponter and Karadeniz, 1985; Hyde et al., 1985; König, 1987; Robinson, 1991; Phan, 1995).

For the variant of the Bree problem concerning shakedown of a closed tube under independent pressure and logarithmic temperature, as stated by Gokhfeld and Cherniavsky (1980), the analytical expressions in Section 4 are the only exact direct solutions available to the authors knowledge.

In the last part of this paper, Section 5, a finite element procedure for shakedown of tubes is proposed and applied to the same closed tube and a restrained tube, under identical loading conditions.

The finite element solution for the restrained tube, which is another variant of the Bree problem is also the first direct solution of this problem, including thick tubes. The interaction diagram obtained here is in good agreement with the specific incremental collapse cyclic loads reported by Hyde et al. (1985) and close to the boundary predicted in the later reference, for the case of thin tubes.

Acknowledgements

This research is partially supported by CNPq (Conselho Nacional de Pesquisa e Desenvolvimento – Brazil) under project 351432/94-9.

References

Bree, J., 1967. Elastic–plastic behaviour of thin tubes subjected to internal pressure and intermittent high heat fluxes with application to fast nuclear-reactor fuel elements. J. Strain Anal. 2 (3), 226–238.

Borges, L.A., Zouain, N., Huespe, A.E., 1996. A nonlinear optimization procedure for limit analysis. Eur. J. Mech. / A Solids 15, 487–512.

Christiansen, E., 1980. Limit analysis in plasticity as a mathematical programming problem. Calcolo 1, 41–65.

Christiansen, E., Andersen, K.D., 1998. Computation of collapse states with von Mises type yield condition. Institut for Matematik og Datalogi, Preprint 18, pp. 1–25.

Fremond, M., Friaa, A., 1982. Les méthodes statique et cinématique en calcul à la rupture et en analyse limite. J. de Mécanique théorique et appliquée 1 (5), 881–905.

Gokhfeld, D.A., Cherniavsky, D.F., 1980. Limit analysis of structures at thermal cycling. Sijthoff and Noordhoof, The Netherlands.

Hyde, T.H., Sahari, B.B., Webster, J.J., 1985. The effect of axial loading and axial restraint on the thermal ratcheting of thin tubes. Int. J. Mech. Sci. 27 (10), 679–692.

Kamenjarzh, J., 1996. Limit Analysis of Solids and Structures, CRC Press Boca Raton.

König, J.A., 1979. On upper bounds to shakedown loads. ZAMM 59, 349–354.

König, J.A., 1987. Shakedown of Elastic–Plastic Structures. Elsevier, Amsterdam.

Lubliner, J., 1990. Plasticity Theory. Macmillan, New York.

Maier, G., 1972. A Shakedown matrix theory allowing for workhardening and second-order geometric effects. In: Sawczuck, A. (Ed.), Foundations of Plasticity. Noordhof, Leyden, pp. 417–433.

Maier, G., 1977. Shakedown analysis. In.: Engineering Plasticity by Mathematical Programming. Proceedings of the NATO Advanced Study Institute. University of Waterloo, Ont., Canada.

Mróz, Z., Weichert, D., Dorosz, (Eds.), 1995. Inelastic Behaviour of structures under variable loads. Kluwer Academic Publishers, London.

Phan, D.C., 1995. Adaptation of spherical and cylindrical vessels to variable internal pressure and temperature. Int. J. Mech. Sci. 37 (7), 783–792.

Polizzotto, C., 1993. On the conditions to prevent plastic shakedown of structures: Part II – the plastic shakedown limit load. J. Appl. Mech., ASME 60, 20–25.

Ponter, A.R.S., Karadeniz, S., 1985. An extended shakedown theory for structures that suffer cyclic thermal loading, Parts 1 and 2. J. Appl. Mech., ASME 52, 877–889.

Robinson, M., 1991. The application of stress resultant theory for incremental collapse of thin tubes. Int. J. Mech. Sci. 33 (10), 805–814.

Rockafellar, T., 1973. Conjugate duality and optimization. Regional Conference Series in Applied Mathematics vol. 16, published by SIAM.

Silveira, J.L.L., Zouain, N., 1997. On extremum principles and algorithms for shakedown analysis. Eur. J. Mech./A Solids 16 (5), 757–778.

Zouain, N., Silveira, J.L.L., 1999. Extremum principles for bounds to shakedown loads. Eur. J. Mech./ A Solids 18 (5), 879–901.

Zouain, N., Silveira, J.L.L., 2000. Variational principles for shakedown analysis. Inelastic Analysis of Structures Variable Loads: Theory and Engineering applications. In: Weichert, D., Maier G., (Eds.), Kluwer, London, accepted for publication.